Graded Types - Part I

Theory and practice of linear types

Dominic Orchard, 24-28th July 2023, SPLV23

UNIVERSITY OF CAMBRIDGE
University of Kent

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```
{n : N}
-> Fin n
-> Fin (n+1)
```

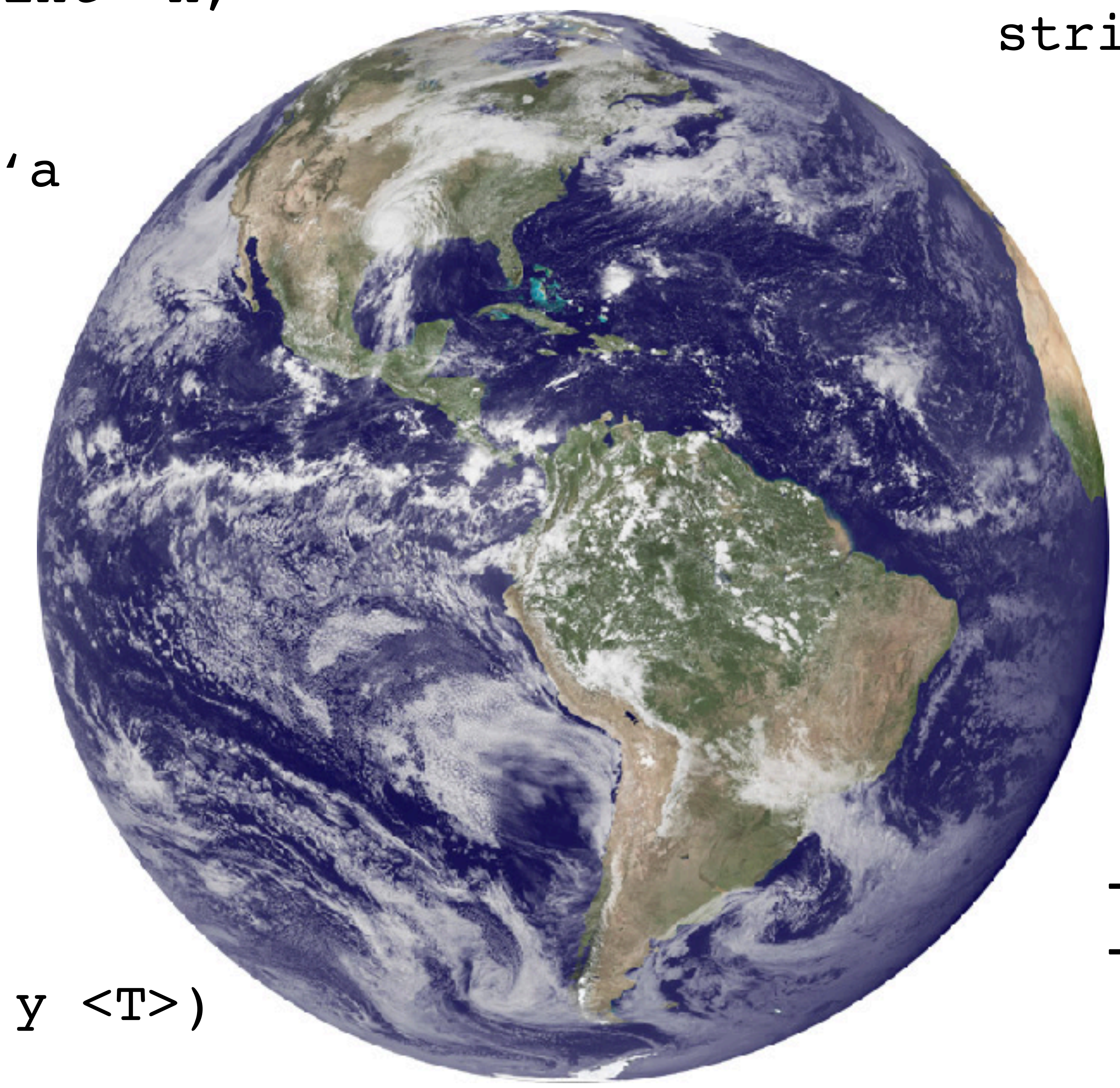
```
factorial(i : u64) -> u64
```

```
int* x;
```

```
string[] args
```

```
‘a * ‘b -> ‘a
```

```
REAL(KIND=8) pi;
```



```
set of 0..10
```

```
Vect n a
-> Vect m a
-> Vect (n+m) a
```

```
   float<‘u>
-> float<‘u ^ 2>
```

```
   [a]
-> (a -> Bool)
-> [a]
```

```
boolean eq(x <T>, y <T>)
```

```
'a => int -> 'a = <handler>
```

```
fib 2/1 :: (integer()) → integer()
```

Types for the “four Rs” of PL design

Reading

- Documentation

‘Riting

- Specification (intention)
- Synthesis

Reasoning

- Guarantee absence of some bugs
- Program properties (see ‘Free Theorems’)

Running

- Optimisations

Dominic A. Orchard:
The four Rs of programming language design. Onward! 2011: 157-162

Impure	Pure
`State Int String`	`String`
`IO String`	

Recall the S4 axioms for modal possibility ◊...

T $$A \to \Diamond A$$

4 $$\Diamond\Diamond A \to \Diamond A$$

K $$\Diamond(A \to B) \to \Diamond A \to \Diamond B$$

Monads as a possibility modality (Benton, Bierman, de Paiva)

Impure	Pure
`State Int String`	`String`
`IO String`	

Impure

Pure

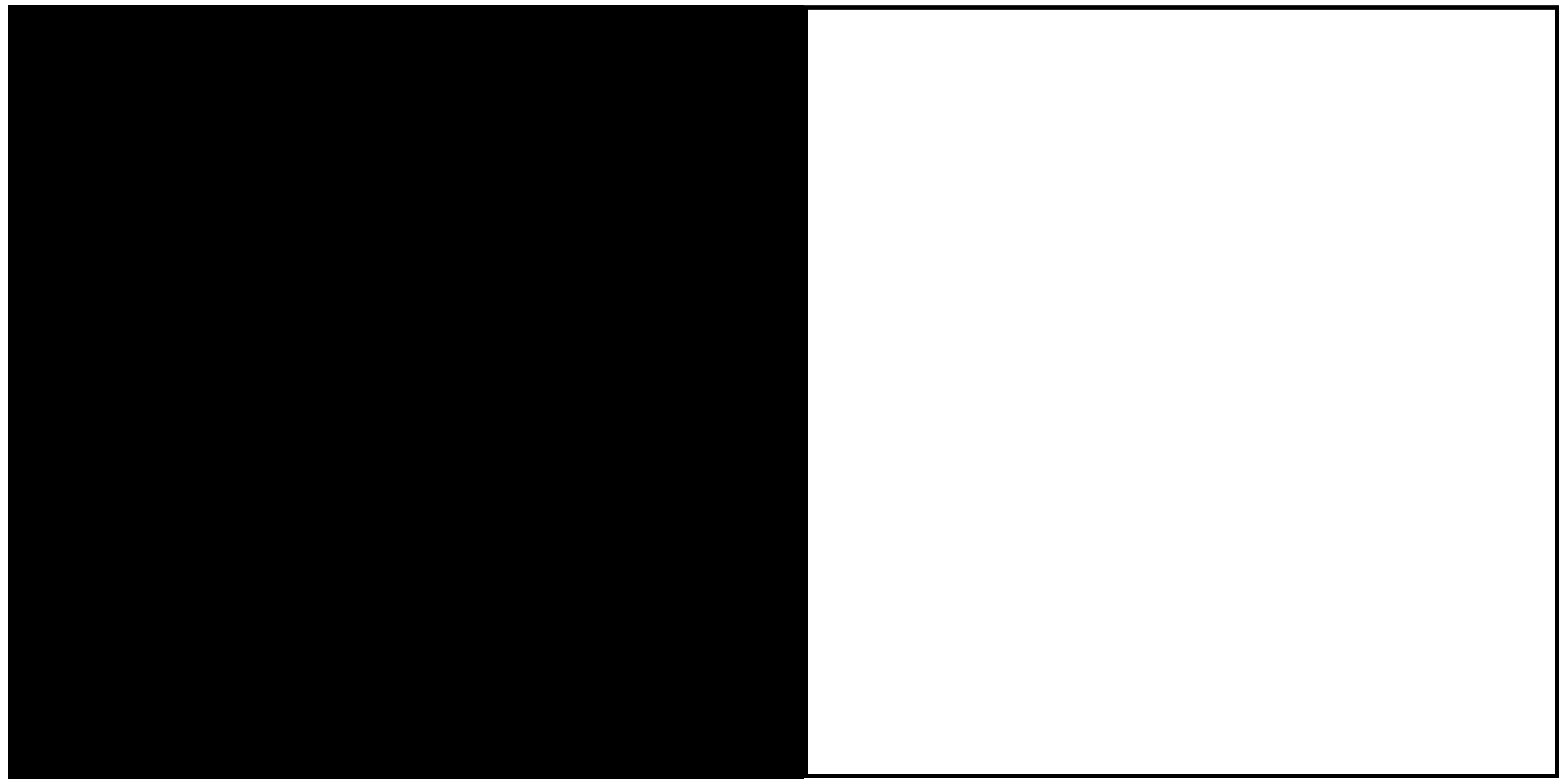
`State Int String`

`String`

Update Write Read Pure

View 1

Modal Type Analysis



$\Box A, \Diamond A$

$!A, \mathrm{M}\, A$

Graded Modal Type Analysis

$!_r A, \mathrm{M}_f\, A$

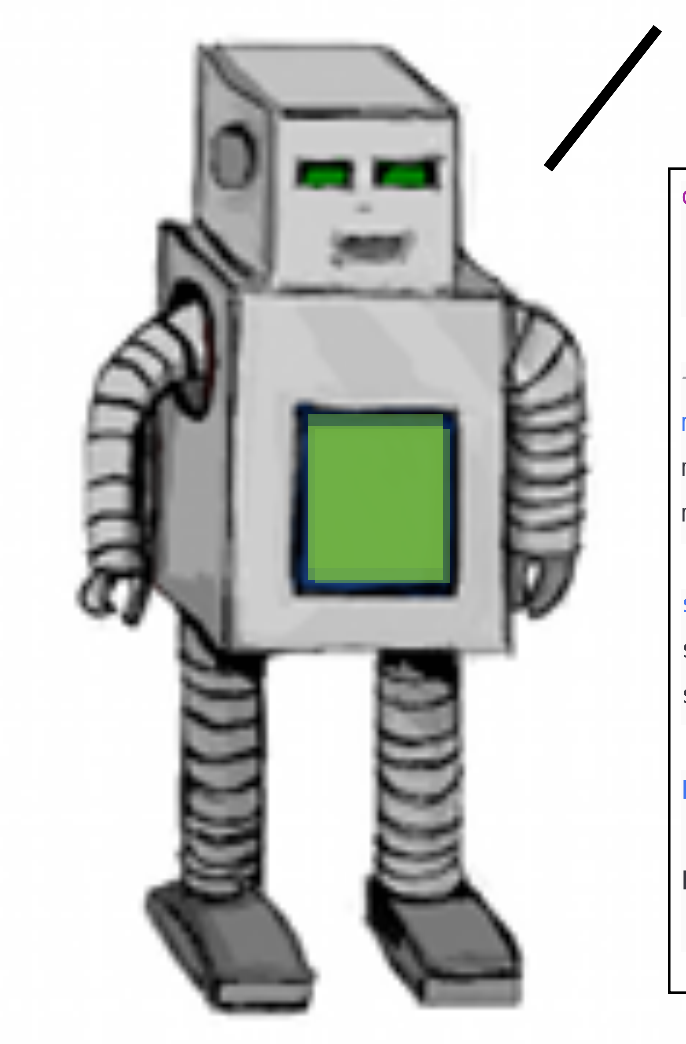
View 2

Intension

Extension

“how”

“what”

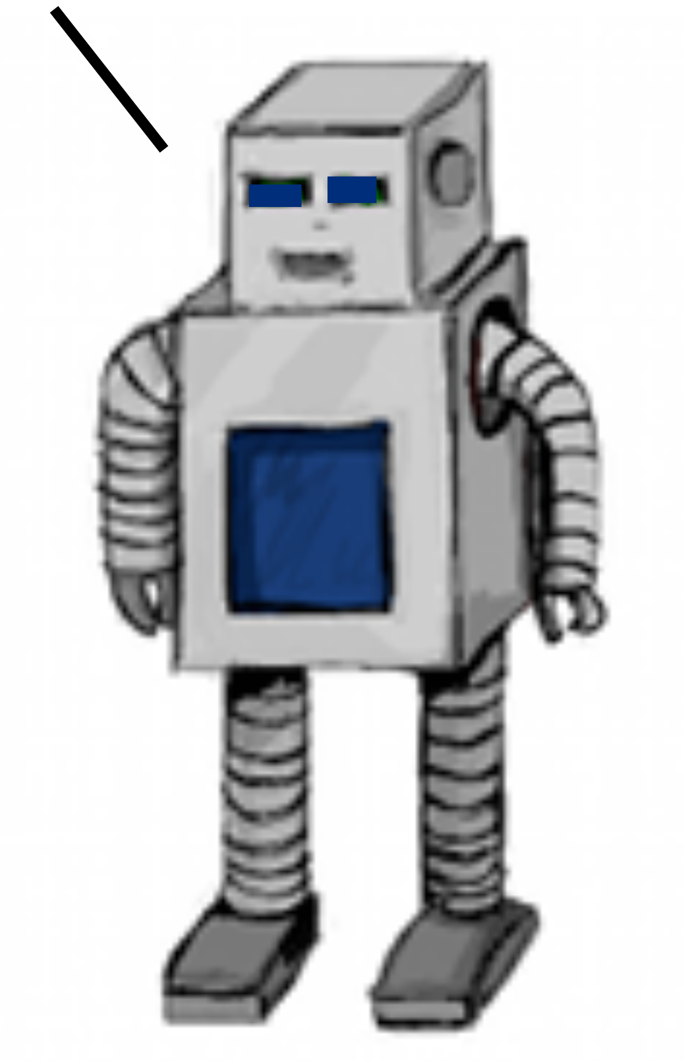


```
data Vec (n : Nat) (a : Type) where
    Nil : Vec 0 a;
    Cons : forall {n : Nat} . a -> Vec n a -> Vec (n+1) a

--- Map function
map : forall {a b : Type, n : Nat} . (a -> b) [n] -> Vec n a -> Vec n b
map [_] Nil = Nil;
map [f] (Cons x xs) = Cons (f x) (map [f] xs)

sequence : forall {n : Nat} . Vec n (() <{Stdout}>) -> () <{Stdout}>
sequence Nil = pure ();
sequence (Cons m xs) = let () <- m in sequence xs

printPerLine : forall {a : Type, n : Nat}
              . Vec n Char -> () <{Stdout}>
printPerLine xs =
 sequence (map [\x -> toStdout (stringAppend (showChar x) ("\n"))] xs)
```



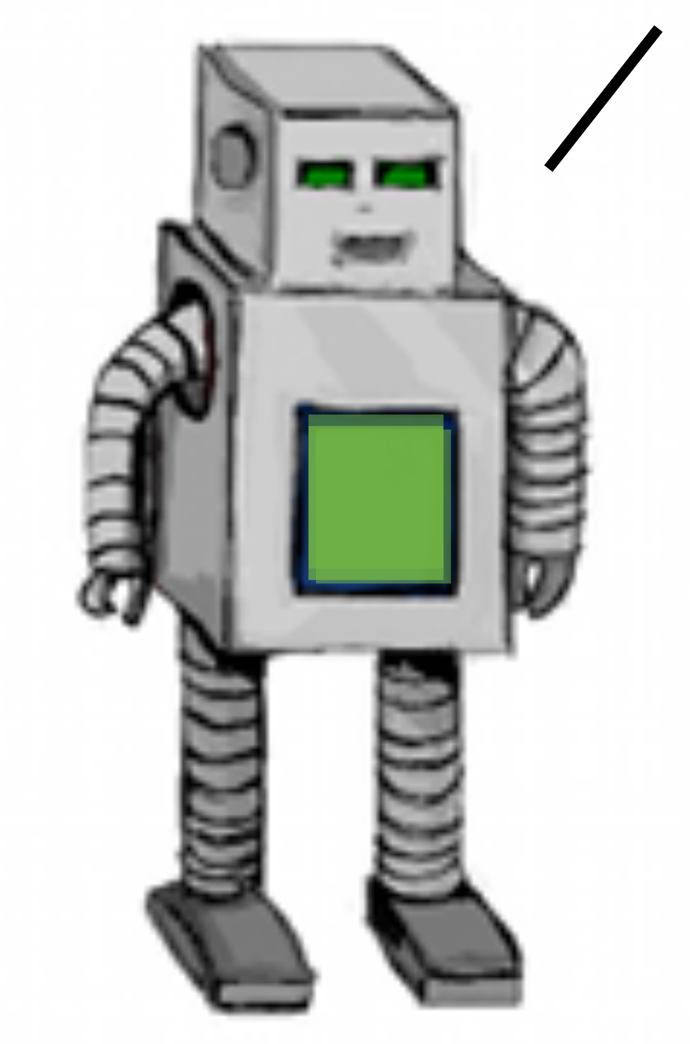
modalities
& grades

types

View 2

Intension

"*how*"



modalities
& grades

This course

how programs use data

how programs depend on context

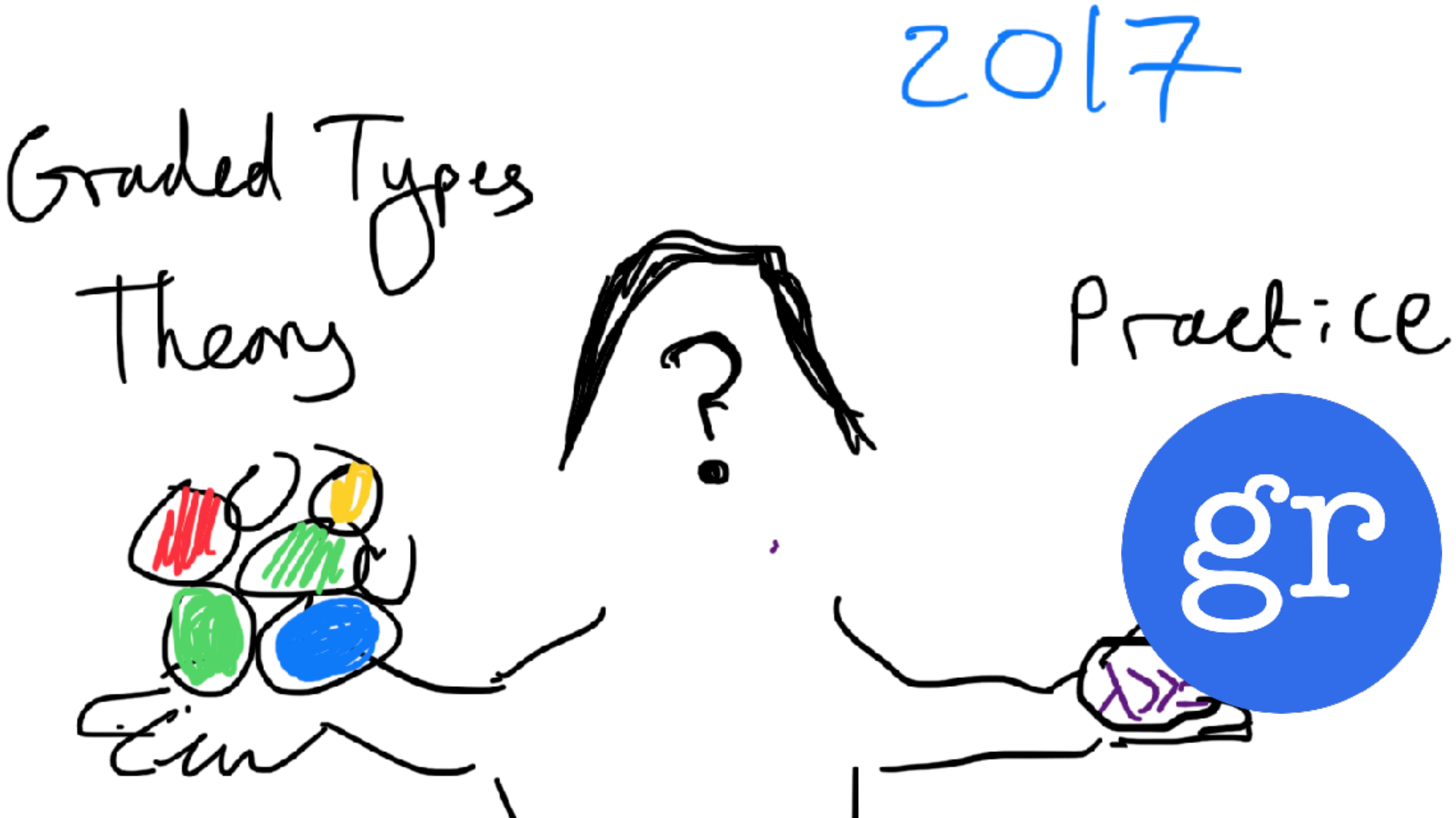
how programs change their context

Our route…

Theory & Practice

Our route…

Theory & Practice




definitely
ask me
ask me
SPLV '23

Session 1 - 2

Learning plan

Learn about linear types

Learn how a type system is formally specified

Specifically: linear types for the *lambda calculus*

See examples of linear programs in Granule

Learn about (a particular flavour of) modalities and graded modalities

Advice: externalise ‘known unknowns’

Things I don’t understand yet	Things I need to get better at	Things I’m getting the hang of

Materials and instructions

https://granule-project.github.io/splv23

Data

Infinitely copiable

Arbitrarily discardable

Universally unconstrained

Data

as a resource

Motivating Example

Problem

(Some) data acts as a resource

Ignoring this leads to bugs!

Solution

Capture resource constraints in types

Do this in a general, extensible way

Linear Logic and the non-linearity (necessity) modality (Girard, 1987)

"Resourceful" interpretation to logic: use exactly once

$$A \vdash A$$

$$A, B \nvdash A$$

$$A \nvdash A \wedge A$$

certainly / always / arbitrarily

but $A, !B \vdash A$

$$!A \vdash A \wedge A$$

Girard, Jean-Yves. "Linear logic." *Theoretical computer science* 50.1 (1987)

Linear types in Granule

(and solving the unsafe files problem)

File handling interface in Granule

like IO a in Haskell

```
openHandle     : IOMode -> String -> Handle <IO>

readChar       : Handle -> (Handle, Char) <IO>

writeChar      : Handle -> Char -> Handle <IO>

isEOF          : Handle -> (Handle, Bool) <IO>

closeHandle    : Handle -> () <IO>
```

Linear lambda calculus

Typing syntax and relation

Church syntax adds a type "signature"

$$t ::= x \mid \lambda(x : A) \,.\, t \mid t_1 \; t_2$$

Type syntax

$$A, B ::= A \multimap B$$

cf Haskell: `t -> t'`

$\mid$ `Int` $\mid$ `Bool` $\mid$ ....

In a full language we'd want more…

Typing lets us relate expressions to types, e.g.

$$\lambda(x : A) \,.\, x : A \multimap A$$

cf.

```
id :: a -> a
id = \x -> x
```

Quick exercise:

Q: What is the type of this lambda term?

$$\lambda(x : A).\lambda(y : B).x$$

A: $A \multimap B \multimap A$

Cf.:

```
const :: a -> b -> a
const x y = x
```

Q: What is the type of this lambda term?

$$\lambda(x : A).y$$

A: *It depends!*

Typing syntax and relation

Typing *judgement* with *assumptions* about variable types

$$y : B \vdash \lambda(x : A) \,.\, y : A \multimap B$$

Assumptions **Term** **Type**

Syntax of assumptions

$$\Gamma ::= \Gamma, x : A \mid \emptyset$$

Typing *judgement* form: $\Gamma \vdash t : A$

Typing rules

Defined inductively

Base case: $\dfrac{}{\textbf{conclusions}}$

Inductive step: $\dfrac{\textbf{premises (inductive hypotheses)}}{\textbf{conclusions}}$

$$\frac{}{x : A \vdash x : A}\ \text{var}$$

A term which is just one variable, has just one assumption

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x \,.\, t : A \multimap B}\ \text{abs}$$

Binding free variables

$$\frac{\Gamma \vdash t : A \multimap B \qquad \Delta \vdash t' : A}{\Gamma, \Delta \vdash t\, t' : B}\ \text{app}$$

Two sub terms have **different** contexts of assumptions

Example

$$\lambda(x : A).\lambda(y : A \to B).y\ x :$$???

Re-ordering in Γ

(abs) takes the "first" assumption:

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x \,.\, t : A \multimap B} \text{ abs}$$

What if we want to lambda bind *y* in the following?

$$\frac{y : A, x : B \vdash t : C}{x : B \vdash \lambda(y : A) \,.\, t : A \multimap C} \text{ ???}$$

✗

Allow reordering in Γ

Can be made explicit:

$$\frac{\Gamma, x : A, y : B, \Delta \vdash t : C}{\Gamma, y : B, x : A, \Delta \vdash t : C} \text{ exchange}$$

$$\frac{\dfrac{y : A, x : B \vdash t : C}{x : B, y : A \vdash t : C} \text{ exchange}}{x : B \vdash \lambda(y : A) \,.\, t : A \multimap C} \text{ abs}$$

✔

A non-example

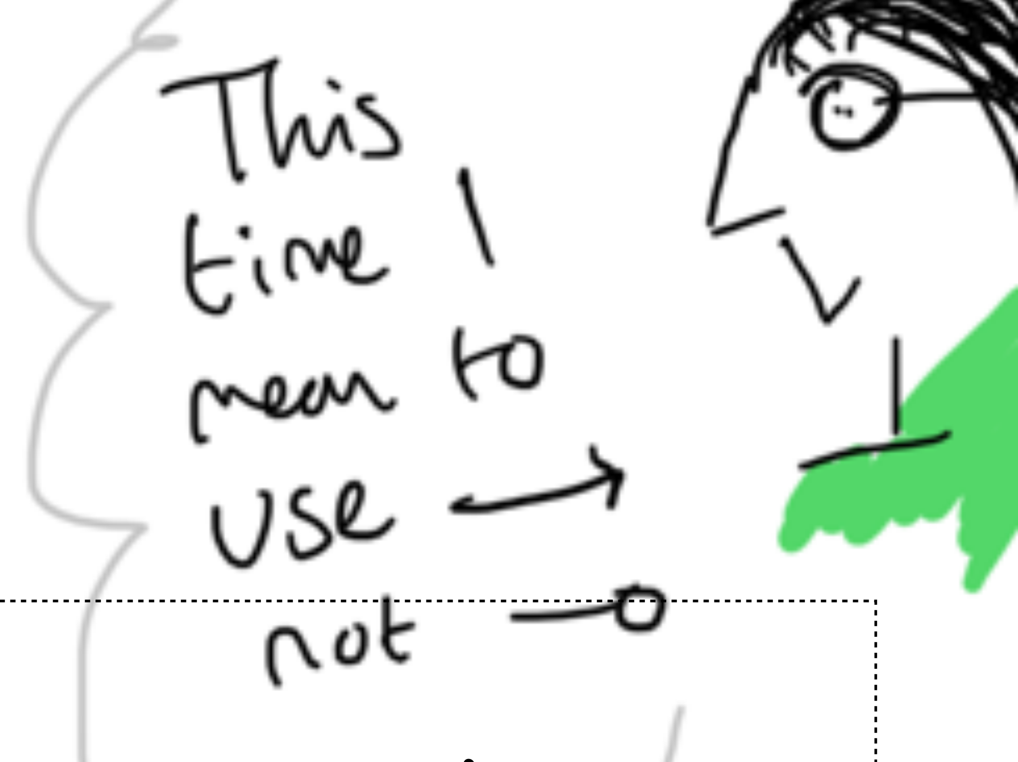
Can't use var rule

$$\dfrac{\dfrac{\dfrac{}{x : A, y : B \vdash x : A}\ ???}{x : A \vdash \lambda(y : B).\, x : B \to A}\ \textbf{abs}}{\emptyset \vdash \lambda(x : A).\, \lambda(y : B).\, x \ : A \to (B \to A)}\ \textbf{abs}$$

Ignoring variable y is dissallowed

Simple typing
(the usual state of affairs…)

Simple typing = Linear typing + weakening + contraction



$$\text{var}\ \frac{}{x : A \vdash x : A} \qquad \text{abs}\ \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \qquad \text{app}\ \frac{\Gamma \vdash t : A \to B \quad \Delta \vdash t' : A}{\Gamma, \Delta \vdash t\, t' : B}$$

Linear λ-calculus typing

$$\text{exchange}\ \frac{\Gamma, x : A, y : B, \Gamma' \vdash t : A}{\Gamma, y : B, x : A, \Gamma' \vdash t : A}$$

$$\text{weaken}\ \frac{\Gamma \vdash t : A}{\Gamma, x : A' \vdash t : A} \qquad \text{contract}\ \frac{\Gamma, y : A', z : A' \vdash t : A}{\Gamma, x : A' \vdash t[x/z][x/y] : A}$$

Ignore variables

Reuse variables

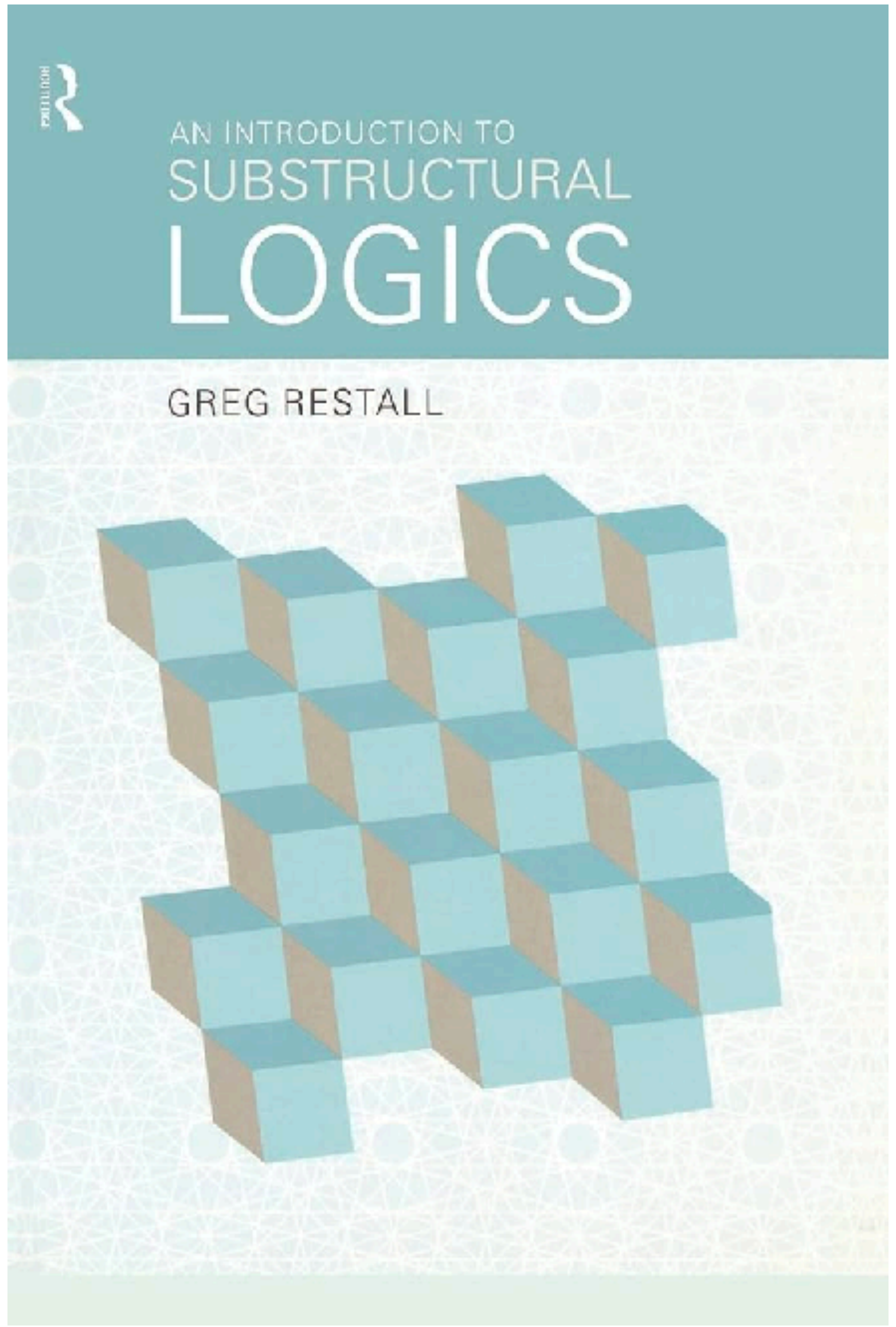
Structural rules

exchange
$$\frac{\Gamma, x : A, y : B, \Gamma' \vdash t : A}{\Gamma, y : B, x : A, \Gamma' \vdash t : A}$$

contract
$$\frac{\Gamma, y : A', z : A' \vdash t : A}{\Gamma, x : A' \vdash t[x/z][x/y] : A}$$

weaken
$$\frac{\Gamma \vdash t : A}{\Gamma, x : A' \vdash t : A}$$

Logics without one or more of these are called "substructural logics"



Weakening example

Couldn't do this in the linear system

$$\dfrac{\dfrac{\dfrac{}{x : A \vdash x : A}\;\text{var}}{x : A, y : B \vdash x : A}\;\textbf{weaken}}{\dfrac{x : A \vdash \lambda(y : B).\, x : B \to A}{\emptyset \vdash \lambda(x : A).\, \lambda(y : B).\, x : A \to (B \to A)}\;\textbf{abs}}\;\textbf{abs}$$

Ignoring variable y

(4) $!A \multimap !!A$

(T) $!A \multimap A$

(K) $!(A \multimap B) \multimap !A \multimap !B$

Behaves like an (S4) modal $\Box$ + some more axioms

! modality — use any number of times

linear logic — use exactly once

Non-linearity modality
in
Granule

(Written postfix `a []` in Granule)

Typing syntax and relation

Extend syntax of types and typing assumptions

$$A, B ::= A \multimap B \mid \Box A$$

Non-linear value of type A

$$\Gamma ::= \emptyset \mid \Gamma, x : A \mid \Gamma, x : [A]$$

Non-linear variable assumption x of type A

(var), (abs), (app) stay the same…

…but we add weakening for non-linear assumptions

$$\frac{\Gamma \vdash t : B}{\Gamma, x : [A] \vdash t : B}\ \text{weak}$$

Linear types + modality

$$\frac{\Gamma_1 \vdash t : A \multimap B \quad \Gamma_2 \vdash t' : A}{\Gamma_1 + \Gamma_2 \vdash t\,t' : B}\ \text{app}$$

Adds contraction

Use anytime we need to combine contexts

$$(\Gamma, x : [A]) + (\Gamma', x : [A]) = (\Gamma + \Gamma'), x : [A]$$

$$\Gamma + (\Gamma', x : A) = (\Gamma + \Gamma'), x : A \ \ \textbf{if}\ x \notin |\Gamma|$$

$$\Gamma, x : A + \Gamma' = (\Gamma + \Gamma'), x : A \ \ \textbf{if}\ x \notin |\Gamma'|$$

Instead of...

$$\frac{\Gamma, x : [A], y : [A] \vdash t : B}{\Gamma, z : [A] \vdash t[z/x][z/y] : B}\ \text{contract}$$

…and syntax + rules for working with non-linearity

Treat a linear variable as non-linear:
(derelection)

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma, x : [A] \vdash t : B}\ \text{der}$$

Non-linear results require non-linear variables
(promotion)

$$\frac{[\Gamma] \vdash t : B}{[\Gamma] \vdash [t] : \Box B}\ \Box_i$$

Composition (substitution) of non-linear value into non-linear variable

$$\frac{\Gamma \vdash t_1 : \Box A \qquad \Delta, x : [A] \vdash t_2 : B}{\Gamma + \Delta \vdash \text{let } [x] = t_1 \text{ in } t_2 : B}\ \Box_e$$

Is this logic "good"?

- "*All logics are invented, some are useful.*"

- Standard probes:

Lemma 1. (Admissibility of substitution). *Let* $\Delta \vdash t' : A$, *then:*

– *(Linear)* *If* $\Gamma, x : A, \Gamma' \vdash t : B$ *then* $\Gamma + \Delta + \Gamma' \vdash [t'/x]t : B$

– *(Modal)* *If* $\Gamma, x : [A], \Gamma' \vdash t : B$ *and* $[\Delta]$ *then* $\Gamma + \Delta + \Gamma' \vdash [t'/x]t : B$

Difficulty of getting this for S4 and ! explained in:

Dag Prawitz. 1965. Natural Deduction: A proof-theoretical study.

Philip Wadler. 1992. There's no substitute for linear logic. In 8th International Workshop on MFPS